15.482 Healthcare Finance
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Unit 4, Part 3: Portfolio Theory

Unit Outline

- Risk & Reward
- The CAPM
- Applications
- Portfolio Theory
- Risk-Adjusted NPV

Portfolio Theory

Motivation

What Is a Portfolio and Why Is It Useful?

- A portfolio is simply a specific combination of securities, usually defined by portfolio weights that sum to 1:

$$\boldsymbol{\omega} = \{\, \omega_1 \,,\, \omega_2 \,, \ldots, \, \omega_n \,\}$$

$$\omega_i = \frac{N_i P_i}{N_1 P_1 + \cdots + N_n P_n}$$

$$1 = \omega_1 + \omega_2 + \cdots + \omega_n$$

- Portfolio weights can sum to 0 (dollar-neutral portfolios), and weights can be positive (long positions) or negative (short positions).
- Assumption: Portfolio weights summarize all relevant information.

Motivation

Example:

- Your investment account of $100,000 consists of three stocks: 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	10%
B	1,000	$60	$60,000	60%
C	750	$40	$30,000	30%
Total			$100,000	100%

Motivation

Example:

- Your broker informs you that you only need to keep $50,000 in your investment account to support the same portfolio of 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C; in other words, you can buy these stocks **on margin**. You withdraw $50,000 to use for other purposes, leaving $50,000 in the account. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	20%
B	1,000	$60	$60,000	120%
C	750	$40	$30,000	60%
Riskless Bond	–$50,000	$1	–$50,000	–100%
Total			$50,000	100%

Motivation

Why Not Pick The Best Asset Instead of Forming a Portfolio?

- We don’t know which asset is best!
- Portfolios provide diversification, reducing unnecessary risks.
- Portfolios can enhance performance by focusing bets.
- Portfolios can customize and manage risk/reward trade-offs.

How Do We Construct a “Good” Portfolio?

- What does “good” mean?
- What characteristics do we care about for a given portfolio?
  - Risk and reward
- Investors like higher expected returns
- Investors dislike risk

Motivation

Assumption:

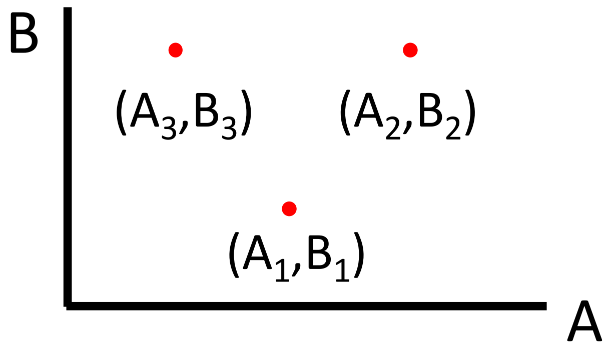
- Investors like high expected returns E[R] but dislike high volatility SD[R]
- Investors care only about the expected return and volatility of their <u>overall portfolio</u>
  - Not individual stocks in the portfolio
  - Investors are generally assumed to be well-diversified

Key questions: How much does an asset contribute to the risk and return of a portfolio, and how can we choose portfolio weights to optimize the risk/reward characteristics of the overall portfolio?

Mean-Variance Analysis

Consider The "Calculus of Consumption"

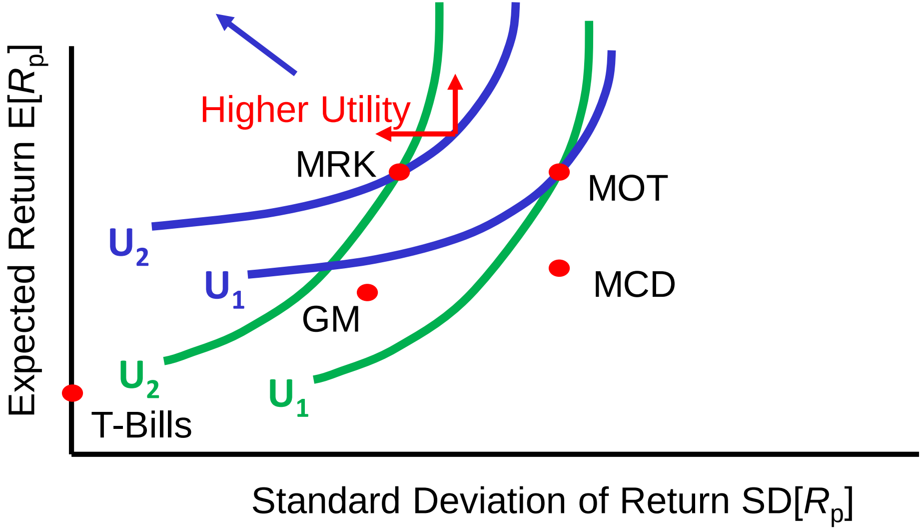
- More is preferred to less (non-satiation, free disposal)
- Transitivity: $X \succ Y$ and $Y \succ Z \Rightarrow X \succ Z$
- Diminishing marginal returns
- These axioms have surprisingly specific implications



Mean-Variance Analysis

Objective:

- Assume investors focus only on the expected return and variance (or standard deviation) of their portfolios: higher expected return is good, higher variance is bad
- Develop a method for constructing optimal portfolios



Mean-Variance Analysis

Basic Properties of Mean and Variance For Individual Returns:

$$
\begin{aligned}
\text{Mean} &= \mathrm{E}[R_i] &&= \mu_i \\
\text{Variance} &= \mathrm{Var}[R_i] &&= \mathrm{E}[(R_i - \mu_i)^2] = \sigma_i^2 \\
\text{Standard Deviation} &= \sqrt{\mathrm{Var}[R_i]} &&= \sigma_i
\end{aligned}
$$

Basic Properties of Mean And Variance For Portfolio Returns:

$$
\boxed{
\begin{aligned}
R_p &= \omega_1 R_1 + \omega_2 R_2 + \cdots + \omega_n R_n \\
\mathrm{E}[R_p] &= \omega_1 \mu_1 + \omega_2 \mu_2 + \cdots + \omega_n \mu_n \\
&= \mu_p \quad \text{(Weighted Average)}
\end{aligned}
}
$$

Mean-Variance Analysis

Variance Is More Complicated:

$$
\begin{aligned}
\mathsf{Var}[R_p] &= \mathsf{E}[(R_p - \mu_p)^2] \\
&= \mathsf{E}\Big[\, \big(\omega_1(R_1 - \mu_1) + \omega_2(R_2 - \mu_2) + \cdots + \\
&\qquad \omega_n(R_n - \mu_n)\, \big)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{E}[\omega_i\omega_j(R_i - \mu_i)(R_j - \mu_j)] &= \omega_i\omega_j\mathsf{Cov}[R_i, R_j] \\
&= \omega_i\omega_j\sigma_{ij} \\
&= \omega_i\omega_j\sigma_i\sigma_j\rho_{ij}
\end{aligned}
$$

Mean-Variance Analysis

Portfolio variance is the weighted sum of all the variances and covariances:

	$\omega_1(R_1-\mu_1)$	$\omega_2(R_2-\mu_2)$	$\cdots$	$\omega_n(R_n-\mu_n)$
$\omega_1(R_1-\mu_1)$	$\omega_1^2\sigma_1^2$	$\omega_1\omega_2\sigma_{12}$	$\cdots$	$\omega_1\omega_n\sigma_{1n}$
$\omega_2(R_2-\mu_2)$	$\omega_2\omega_1\sigma_{21}$	$\omega_2^2\sigma_2^2$	$\cdots$	$\omega_2\omega_n\sigma_{2n}$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\omega_n(R_n-\mu_n)$	$\omega_n\omega_1\sigma_{n1}$	$\omega_n\omega_2\sigma_{n2}$	$\cdots$	$\omega_n^2\sigma_n^2$

- There are n variances, and $n^2 - n$ covariances
- Positive covariances increase portfolio variance; negative covariances decrease portfolio variance (diversification)

Mean-Variance Analysis

Portfolio variance is the weighted sum of all the variances and covariances:

$$\text{Var}[R_p] = \text{Var}[\omega'\mathbf{R}] = \omega'\Sigma\omega = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}}_{\Sigma} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}$$

- Covariances are central to portfolio theory
- They determine how much de-risking is possible
- The smaller (or more negative) the covariances, the greater the risk reduction

Mean-Variance Analysis

Consider The Special Case of Two Assets:

$$R_p = \omega_a R_a + \omega_b R_b$$

$$\begin{aligned}
\mathsf{E}[R_p] &= \omega_a \mu_a + \omega_b \mu_b \\
\mathsf{Var}[R_p] &= \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \mathsf{Cov}[R_a, R_b] \\
&= \omega_a^2 \sigma_a^2 + \omega_b^2 \sigma_b^2 + 2\omega_a \omega_b \sigma_a \sigma_b \rho_{ab}
\end{aligned}$$

$$\text{Because} \quad \rho_{ab} \equiv \frac{\mathsf{Cov}[R_a, R_b]}{\sigma_a \sigma_b}$$

$$\mathsf{Cov}[R_a, R_b] = \sigma_a \sigma_b \rho_{ab}$$

- As correlation increases, overall portfolio variance increases

Mean-Variance Analysis

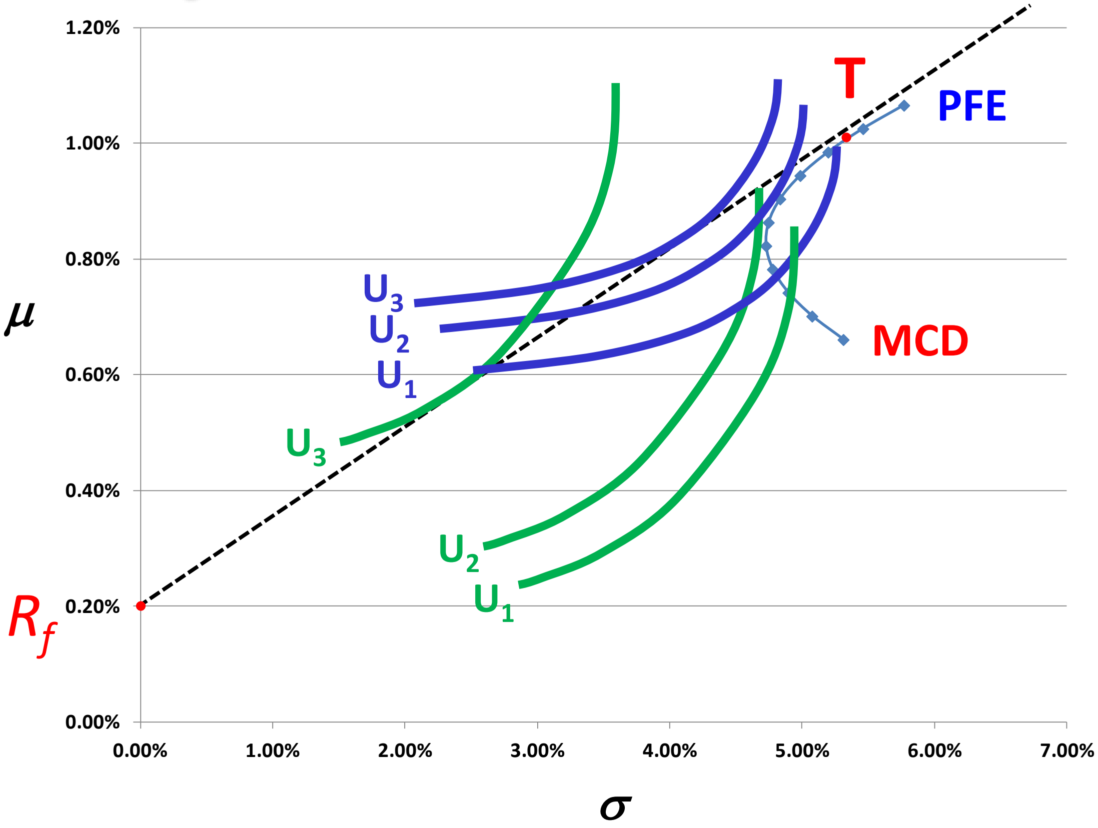
Example: From 2011:1– 2-16:12, Pfizer had an average monthly return of 1.07% and a SD of 4.36%. McDonald's had an average return of 0.66% and a SD of 3.73%. Their return correlation is 44.0%. How would a portfolio of the two stocks perform?

$$\mathrm{E}[R_p] = \omega_{\mathrm{PFE}}(0.0107) + \omega_{\mathrm{MCD}}(0.0066)$$

$$\mathrm{Var}[R_p] = \omega^2_{\mathrm{PFE}}(0.0436)^2 + \omega^2_{\mathrm{MCD}}(0.0373)^2 + 2\omega_{\mathrm{PFE}}\omega_{\mathrm{MCD}}(0.440)(0.0436)(0.0373)$$

Mean-Variance Analysis

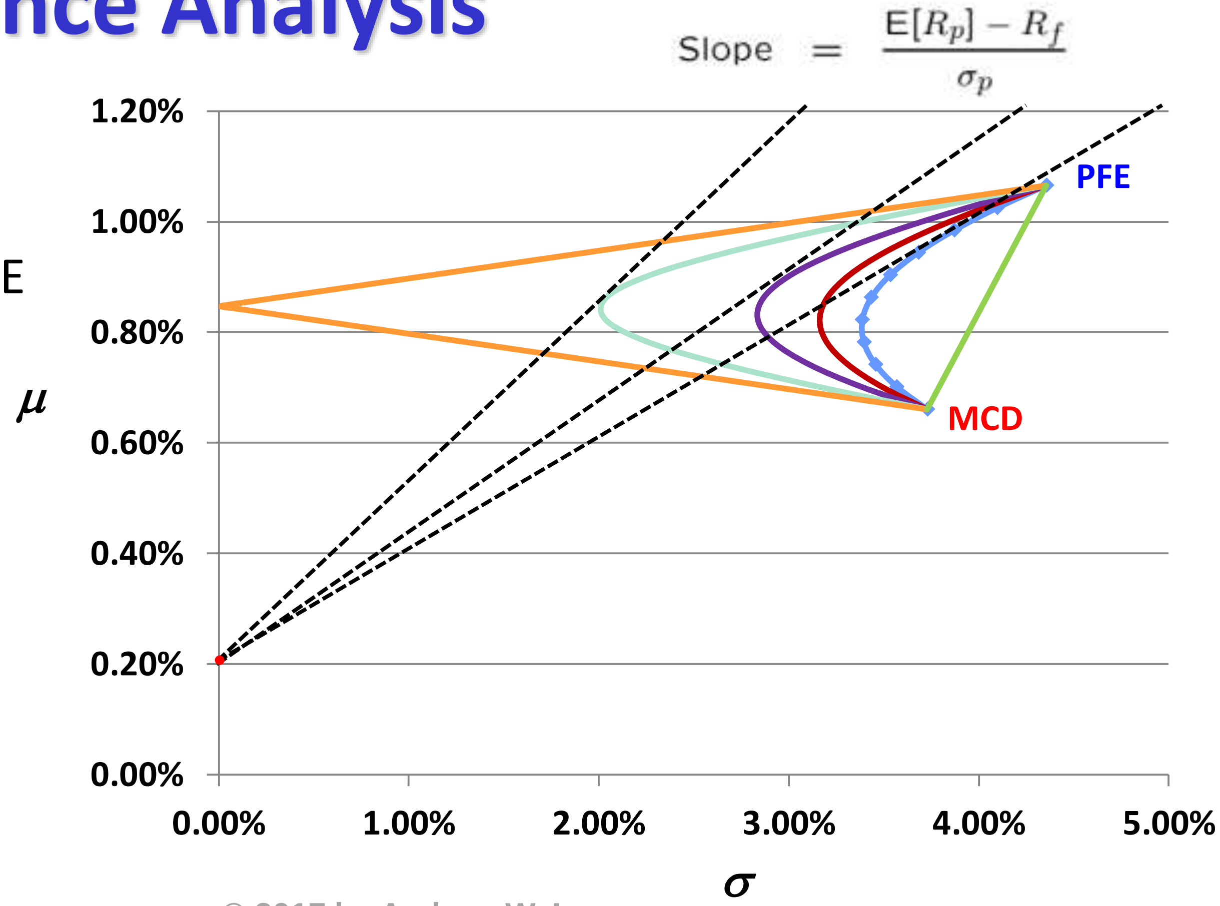
w_{PFE}	$E[R_p]$	$Var[R_p]$	$SD[R_p]$
0%	0.66%	0.002820	5.31%
10%	0.70%	0.002574	5.07%
20%	0.74%	0.002395	4.89%
30%	0.78%	0.002281	4.78%
40%	0.82%	0.002233	4.73%
50%	0.86%	0.002251	4.74%
60%	0.90%	0.002335	4.83%
70%	0.94%	0.002484	4.98%
80%	0.99%	0.002699	5.20%
90%	1.03%	0.002980	5.46%
100%	1.07%	0.003327	5.77%



Mean-Variance Analysis

What if correlation between MCD and PFE were:

- ρ = 25%?
- ρ = 0%?
- ρ = –50%?
- ρ = –100%?
- ρ = +100%?



Mean-Variance Analysis

In General:

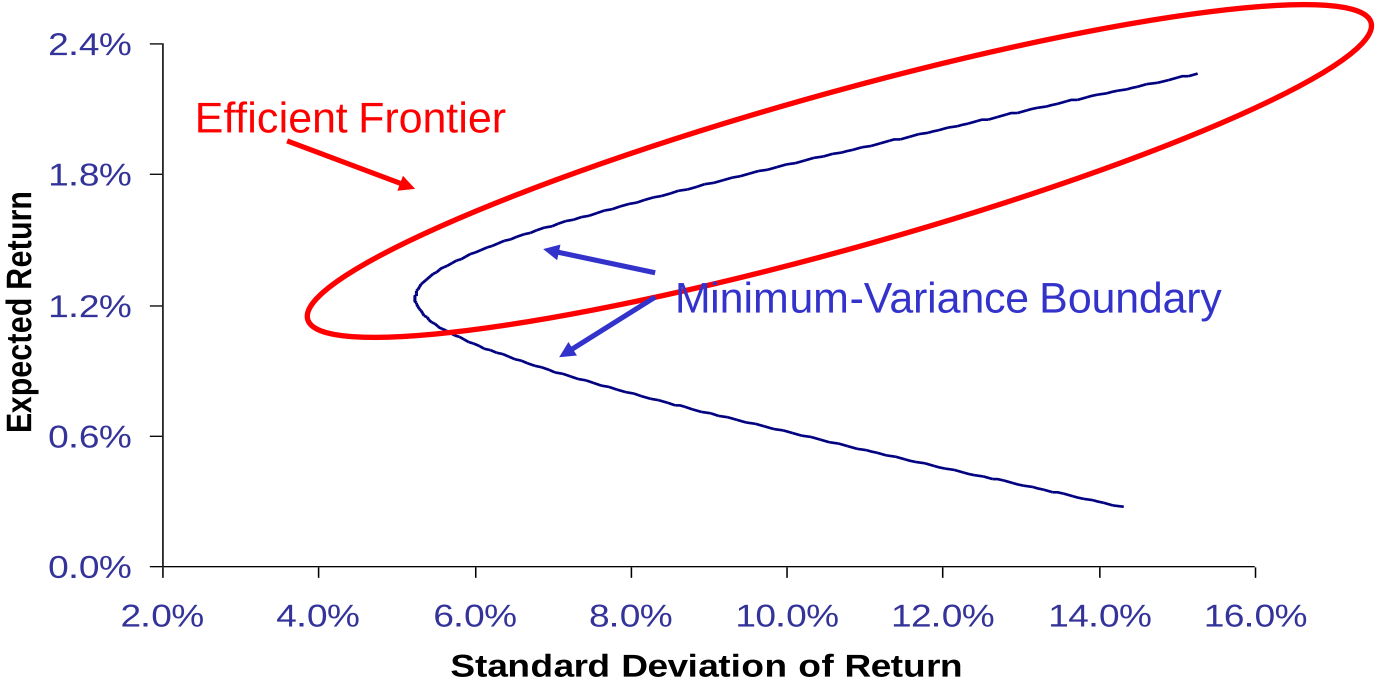
$$\begin{aligned} \mathsf{E}[R_p] &= \mu_p = \omega_1\mu_1 + \cdots + \omega_n\mu_n &&= \boldsymbol{\omega}'\boldsymbol{\mu} \\ \mathsf{Var}[R_p] &= \sigma_p^2 = \textstyle\sum_{i=1}^n \sum_{j=1}^n \omega_i\omega_j \mathsf{Cov}[R_i, R_j] &&= \boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} \end{aligned}$$

Observations:

- E[R_p] is a weighted average of all the assets' expected returns
- SD[R_p] is smaller if assets' correlations are lower. It is **less than** a weighted average of the assets' standard deviations (unless perfect correlation)
- The graph of portfolio mean/SD is nonlinear; find the "best" $\{\omega_i\}$ (quadratic optimization)
- If we combine T-Bills with any risky stock, portfolios plot along a straight line

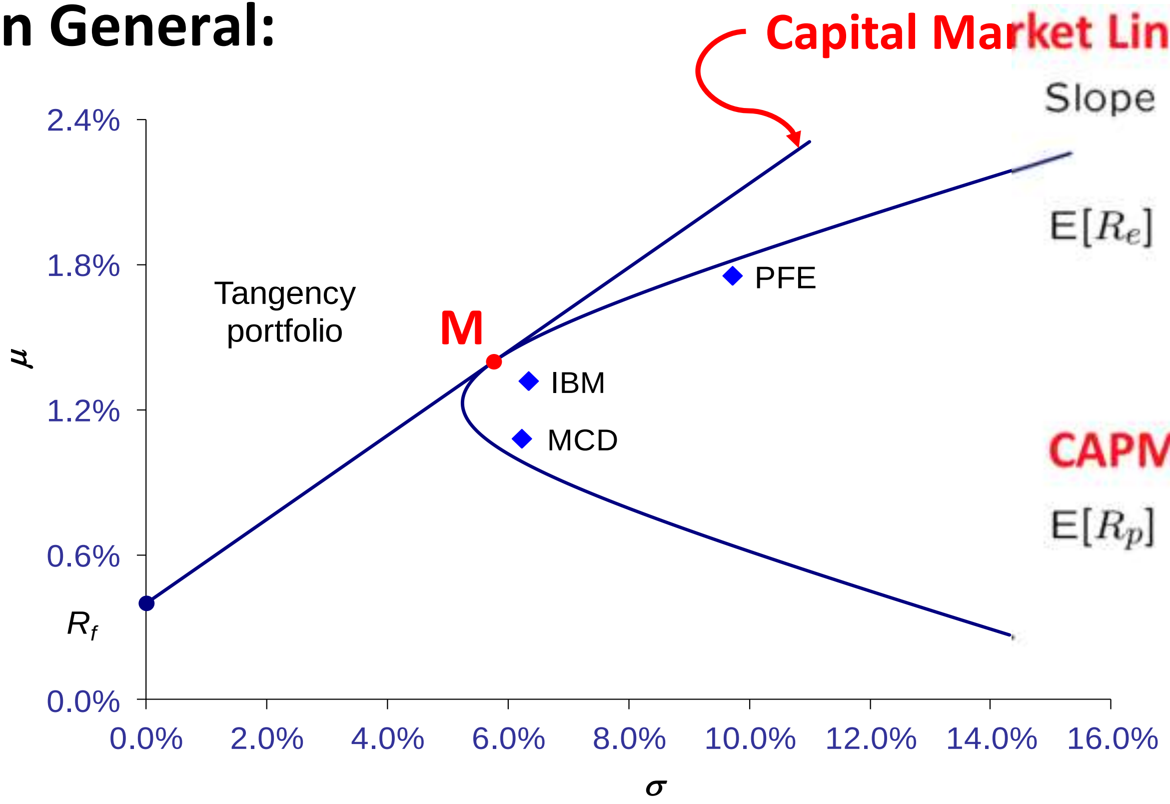
Mean-Variance Analysis

In General:



Mean-Variance Analysis

In General:



$$\text{Slope} = \frac{\mathrm{E}[R_m] - R_f}{\sigma_m}$$

$$\mathrm{E}[R_e] = R_f + \frac{\mathrm{E}[R_m] - R_f}{\sigma_m}\sigma_e$$

$$= R_f + \frac{\sigma_e}{\sigma_m}(\mathrm{E}[R_m] - R_f)$$

CAPM:

$$\mathrm{E}[R_p] = R_f + \beta_p(\mathrm{E}[R_m] - R_f)$$

$$\beta_p \equiv \frac{\mathrm{Cov}[R_p, R_m]}{\mathrm{Var}[R_m]}$$

Mean-Variance Analysis

Special Case #1:

- Consider an equally weighted portfolio:

$$\omega_i = \frac{1}{n} \;, \quad i = 1, \ldots, n$$

$$\begin{aligned} \mathrm{Var}[R_p] &= \sum_{i=1}^{n} \frac{\sigma_i^2}{n^2} + \frac{1}{n^2} \sum_{i \neq j} \mathrm{Cov}[R_i, R_j] \\ &= \frac{1}{n} \times \text{Average Variance} + \frac{n-1}{n} \times \text{Average Covariance} \\ &\approx \text{Average Covariance} \end{aligned}$$

- Managing covariance is more important for large *n*

Mean-Variance Analysis

Special Case #2: equicorrelated returns

- Consider an equal-weighted portfolio of n assets with the same mean μ and variance σ^2 and identical pairwise correlation ρ:

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix} \quad , \quad \rho > -\frac{1}{n-1}$$

$$\text{Sharpe Ratio} = \frac{\mu}{\sigma}\frac{1}{\sqrt{\rho + (1-\rho)/n}} = \text{SR}\,\theta$$

Mean-Variance Analysis

Special Case #2: equicorrelated returns

$$\theta = \frac{1}{\sqrt{\rho + (1-\rho)/n}}$$

- As n increases, θ also increases, with a limit of $1/\sqrt{\rho}$ when $\rho \neq 0$
- If $\rho = 0$ then $\theta = \sqrt{n}$
- Tremendous benefits of uncorrelated assets!

Correlation ρ	Number of Assets *n*											
	1	2	5	10	20	30	40	50	100	150	200	∞
90%	1.00	1.03	1.04	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05
80%	1.00	1.05	1.09	1.10	1.11	1.11	1.11	1.12	1.12	1.12	1.12	1.12
70%	1.00	1.08	1.15	1.17	1.18	1.19	1.19	1.19	1.19	1.19	1.19	1.20
60%	1.00	1.12	1.21	1.25	1.27	1.28	1.28	1.28	1.29	1.29	1.29	1.29
50%	1.00	1.15	1.29	1.35	1.38	1.39	1.40	1.40	1.41	1.41	1.41	1.41
40%	1.00	1.20	1.39	1.47	1.52	1.54	1.55	1.56	1.57	1.57	1.58	1.58
30%	1.00	1.24	1.51	1.64	1.73	1.76	1.77	1.78	1.80	1.81	1.82	1.83
20%	1.00	1.29	1.67	1.89	2.04	2.10	2.13	2.15	2.19	2.21	2.21	2.24
10%	1.00	1.35	1.89	2.29	2.63	2.77	2.86	2.91	3.03	3.07	3.09	3.16
0%	**1.00**	**1.41**	**2.24**	**3.16**	**4.47**	**5.48**	**6.32**	**7.07**	**10.00**	**12.25**	**14.14**	**∞**
-10%	1.00	1.49	2.89	10.00								
-20%	1.00	1.58	5.00									
-30%	1.00	1.69										
-40%	1.00	1.83										
-50%	1.00	2.00										
-60%	1.00	2.24										
-70%	1.00	2.58										
-80%	1.00	3.16										
-90%	1.00	4.47										

Risk-Adjusted NPV

Review of NPV Rule

- If companies want to increase their current market value, they should take only projects with positive NPV

$$\text{NPV} = \text{CF}_0 + \frac{\text{CF}_1}{(1+r)} + \frac{\text{CF}_2}{(1+r)^2} + \cdots + \frac{\text{CF}_T}{(1+r)^T}$$

- Investment criteria:
  - For a single project, take it only if it is NPV positive
  - For many independent projects, take all those with positive NPV
  - For mutually exclusive projects, take the one with positive and highest NPV
  - For dependent projects, take the combination with the highest overall NPV

Capital Budgeting with Risk

- Risk attitudes will affect your decision
- The Wisdom of Crowds should provide objective valuations that account for risk
- First: compute market valuation
  - Compute expected cash flows
  - Discount by risk-adjusted cost of capital (e.g., CAPM)
- Second: consider unique preferences, constraints, objectives, and types of risk/uncertainty

Risk-Adjusted NPV (rNPV)

Given a Sequence of Cash Flows $\{CF_1, CF_2, \ldots, CF_n\}$:

$$\text{rNPV} = \text{CF}_0 + \frac{\text{E}[\text{CF}_1]}{(1+R)} + \frac{\text{E}[\text{CF}_2]}{(1+R)^2} + \cdots + \frac{\text{E}[\text{CF}_n]}{(1+R)^n}$$

- Investment criteria:
  - For a single project, take it only if it is rNPV positive
  - For many independent projects, take all those with positive rNPV
  - For mutually exclusive projects, take the one with positive and highest rNPV
  - For dependent projects, take the combination with the highest overall rNPV